Discontinuity growth of interval exchange maps

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Abstract

For an interval exchange map f, the number of discontinuities $d(f^n)$ either exhibits linear growth or is bounded independently of n. This dichotomy is used to prove that the group $\mathcal{E}$ of interval exchanges does not contain distortion elements, giving examples of groups that do not act faithfully via interval exchanges. As a further application of this dichotomy, a classification of centralizers in $\mathcal{E}$ is given. This classification is used to show that $\mathrm{Aut}(\mathcal{E}) \cong \mathcal{E} \rtimes \mathbb{Z}/2\mathbb{Z}$.

1 Introduction

An interval exchange transformation is a map $\mathbb{T}^1 \to \mathbb{T}^1$ defined by a partition of the circle into a finite union of half-open intervals and a rearrangement of these intervals by translation. See Figure 1 for a graphical example.

The dynamics of interval exchanges were first studied in the late seventies by Keane[**2**],[**3**], Rauzy[**9**], Veech[**10**], and others. This initial stage of research culminated in the independent proofs by Masur[**7**] and Veech[**11**] that almost every interval exchange is uniquely ergodic. See the recent survey of Viana[**12**] for a unified presentation of these results. The current study of interval exchanges is closely related to dynamics on the moduli space of translation surfaces; an introduction to this topic and its connection to interval exchanges is found in a survey of Zorich[**14**].

To precisely define an interval exchange, let $\pi \in \Sigma_n$ be a permutation and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a vector in the simplex

$$\Lambda_n = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n : \lambda_i > 0 \text{ and } \sum \lambda_i = 1\}.$$

The vector λ induces a partition of $\mathbb{T}^1 \cong [0, 1)$ into intervals of length λ_j:

$$I_j = [\beta_{j-1}, \beta_j) \quad 1 \le j \le n,$$

$$\text{where } \beta_0 = 0, \quad \beta_j = \sum_{i=1}^{j} \lambda_i \text{ for } 1 \le j \le n.$$

The interval exchange $f_{(\pi,\lambda)}$ reorders the I_j by translation, such that their indicies are ordered by $\pi^{-1}(1), \pi^{-1}(2), \ldots, \pi^{-1}(n)$. Consequently, $f_{(\pi,\lambda)}$ is defined by the formula

$$f_{(\pi,\lambda)}(x) = x - \left(\sum_{i<j} \lambda_i \right) + \left(\sum_{\pi(i)<\pi(j)} \lambda_i \right) = x + \omega_j, \text{ for } x \in I_j.$$

The vector $\omega(f) = (\omega_1, \ldots, \omega_n)$ is called the translation vector of $f_{(\pi,\lambda)}$.

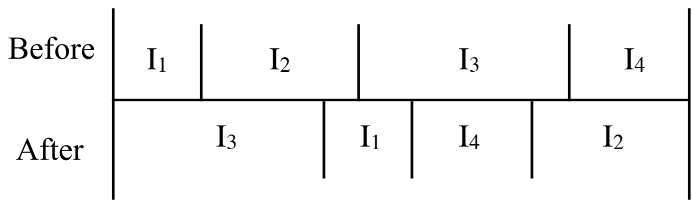

Figure 1: An interval exchange with $\pi = (2, 4, 1, 3)$

Let $d(f)$ denote the number of discontinuity points of $f : \mathbb{T}^1 \to \mathbb{T}^1$, where $\mathbb{T}^1$ is endowed with its standard topology. If $d(f) = k$, then it is easy to see that for iterates f^n, the discontinuity number $d(f^n)$ is bounded above by $k|n|$. It is possible for $d(f^n)$ to have linear growth at a rate which is strictly less than the maximum $d(f)$. For example, the map in Figure 2 has three discontinuities, but iteration will suggest that $d(f^n) \sim 2n$. Additionally, it is possible for $d(f^n)$ to be bounded independently of n. For example, a restricted rotation $r_{\alpha,\beta}$, as defined by Figure 3, satisfies $d(r^n_{\alpha,\beta}) \le 3$ for all $n \in \mathbb{Z}$. The key result of this paper is the observation that no intermediate growth rate may occur.

Theorem 1.1. *For any interval exchange f, either $d(f^n)$ exhibits linear growth or $d(f^n)$ is bounded independently of n.*

This theorem is a simplified statement of Proposition 2.3. The linear growth case is generic; for instance, given any irreducible permutation π which permutes three or more intervals, $f_{(\pi,\lambda)}$ has linear discontinuity growth if the boundary points between intervals satisfy the *infinite distinct orbit condition* [**2**]. This condition is satisfied for the full measure set of $\lambda \in \Lambda_n$ that have rationally independent partition lengths $\{\lambda_i\}$.

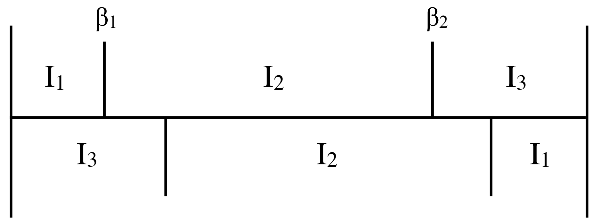

Figure 2: $d(f^n)$ exhibits linear growth (β_1 and β_2 independent over $\mathbb{Q}$)

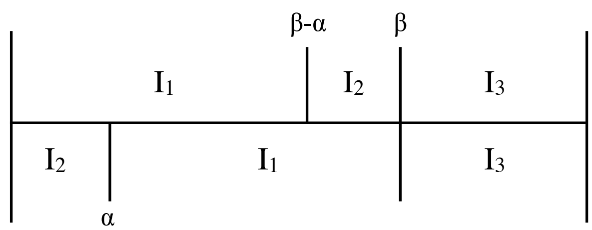

Figure 3: The restricted rotation $r_{\alpha,\beta} : \pi = (2, 1, 3)$, $\lambda = (\beta - \alpha, \alpha, 1 - \beta)$

This raises the question of what may be said about the interval exchanges f for which $d(f^n)$ is bounded. A result of Li [**4**] stated in Section 3 asserts that under certain additional conditions, the only such topologically minimal examples are maps conjugate to an irrational rotation.

By only assuming that $d(f^n)$ is bounded, it is still possible to give a complete description of the interval exchanges with bounded discontinuity growth. For $\gamma \in \mathbb{R}/\mathbb{Z} \cong [0, 1)$, let r_γ denote the rotation $x \mapsto x + \gamma$, which is represented by the data $\pi = (2, 1), \lambda = (1 - \gamma, \gamma)$. An interval exchange is a *restricted rotation* if it is conjugate by some r_γ to some $r_{\alpha,\beta}$. The *support* of an interval exchange f is the complement of the set of fixed points $\mathrm{Fix}(f)$. The following classification of interval exchanges with bounded discontinuity growth is proved in Section 3.

Theorem 1.2. *Let f be an infinite-order interval exchange transformation for which $d(f^n)$ is bounded. Then for some $k \geq 1$, f^k is conjugate to a product of infinite-order restricted rotations with pairwise disjoint supports.*

The discontinuity growth dichotomy and the subsequent classification of maps with bounded discontinuity growth may be applied to the study of interval exchange group actions. The set $\mathcal{E}$ of all interval exchange transformations

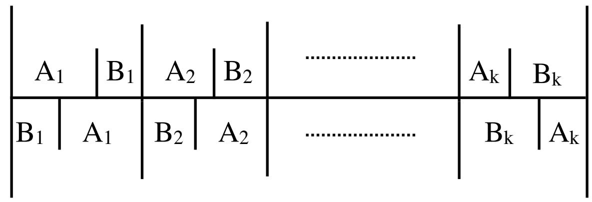

Figure 4: A product of restricted rotations

on $\mathbb{T}^1$ forms a group under composition. For a group G, an *interval exchange action* of G is a group homomorphism $G \to \mathcal{E}$. Such an action is *faithful* if this homomorphism is injective, in which case the image is a subgroup of $\mathcal{E}$ isomorphic to G.

Let G be a finitely generated group, and let $S = \{g_1, \ldots, g_n\}$ be a set of generators. An element $f \in G$ is a *distortion element* if f has infinite order and

$$\liminf_{n\to\infty} \frac{|f^n|_S}{n} = 0,$$

where $|\cdot|_S$ denotes the minimal word length in terms of the generators and their inverses. For example, the central elements of the discrete Heisenberg group are distortion elements. In general, if G is not finitely generated, an element $f \in G$ is said to be a distortion element in G if it is a distortion element in some finitely generated subgroup of G.

Theorem 1.3. *The group $\mathcal{E}$ contains no distortion elements.*

A proof of this result is given in Section 4. The main consequence of this theorem is that any group G containing a distortion element has no faithful interval exchange actions. A particularly interesting case is the following, which is analogous to a result of Witte **[13]** for group actions $SL(n,\mathbb{Z}) \to \text{Homeo}_+(S^1)$ for $n \geq 3$.

Corollary 1.4. *Suppose Γ is a non-uniform irreducible lattice in a semisimple Lie group G with $\mathbb{R}$-rank ≥ 2. Suppose further that G is connected, with finite center and no nontrivial compact factors. Then any interval exchange action $\Gamma \to \mathcal{E}$ has finite image.*

For example, the lattices $SL(n,\mathbb{Z}), n \geq 3$, satisfy the above hypotheses; consequently, they do not act faithfully via interval exchange maps. This corollary follows from a theorem of Lubotzky, Moses, and Raghunathan **[5]**

which states that lattices satisfying the above conditions contain distortion elements (in fact, elements with logarithmic word growth) and a theorem of Margulis **[6]** which states that any irreducible lattice in a semisimple Lie group of $\mathbb{R}$-rank ≥ 2 is almost simple; i.e., any normal subgroup of such a lattice is finite or has finite index.

A further application of the discontinuity growth dichotomy is the complete classification of centralizers in the group $\mathcal{E}$, which is developed in Section 5. This classification relies on analyzing the centralizer $C(f)$ in three cases that are distinguished by dynamical characteristics.

Proposition 1.5. *Let f be an interval exchange transformation.*

(i) *f has periodic points if and only if $C(f)$ contains a subgroup isomorphic to $\mathcal{E}$.*

(ii) *If f is minimal and $d(f^n)$ is bounded, then $C(f)$ is virtually abelian and contains a subgroup isomorphic to $\mathbb{R}/\mathbb{Z}$.*

(iii) *If f is minimal and $d(f^n)$ has linear growth, then $C(f)$ is virtually cyclic.*

Minimality here refers to topological minimality: every orbit of f is dense in $\mathbb{T}^1$. The three parts of this result are restated and proved separately as Corollary 5.8, Proposition 5.2, and Proposition 5.3, respectively. These cases may be combined to give a general description of centralizers in $\mathcal{E}$; this is stated and proved as Theorem 5.7.

The classification of centralizers in $\mathcal{E}$ may be used to investigate the automorphism group $\mathrm{Aut}(\mathcal{E})$. Since $\mathcal{E}$ has trivial center, the inner automorphism group is isomorphic to $\mathcal{E}$. A further automorphism is induced by switching the orientation of the circle $\mathbb{T}^1$. More precisely, let $T : \mathbb{T}^1 \to \mathbb{T}^1$ be defined by $T(x) = -x$. For $f \in \mathcal{E}$, $T^{-1} f T$ is still an invertible piecewise translation, but it is now continuous from the left. Let Ψ_T be the automorphism of $\mathcal{E}$ defined by conjugation with T followed by the natural isomorphism from the group of left-continuous interval exchanges to the right-continuous interval exchange group $\mathcal{E}$.

The automorphism Ψ_T is of interest because it is not an inner automorphism. One way to see this is through the homomorphism $\phi : \mathcal{E} \to \mathbb{R} \wedge_{\mathbb{Q}} \mathbb{R}$, defined by

$$\phi(f_{(\pi,\lambda)}) = \sum_{i=1}^{n} \lambda_i \wedge_{\mathbb{Q}} \omega_i.$$

See Arnoux [**1**] for a discussion of the properties of this map. The rotation r_α is defined by the data $\pi = (2, 1)$, $\lambda = (1 - \alpha, \alpha)$, and it may be checked that $\phi(r_\alpha) = 1 \wedge \alpha$. Any inner automorphism preserves ϕ, but the action of Ψ_T changes the sign of the scissors invariant; for instance,

$$\phi(\Psi_T(r_\alpha)) = \phi(r_{-\alpha}) = -\phi(r_\alpha).$$

Ψ_T is of further interest because it represents the only nontrivial class of outer automorphisms. Section 6 presents a proof of the following result.

Theorem 1.6. *$Aut(\mathcal{E}) = Inn(\mathcal{E}) \rtimes \langle \Psi_T \rangle \cong \mathcal{E} \rtimes \mathbb{Z}/2\mathbb{Z}$.*

Note that the inner automorphisms and the automorphism Ψ_T act via conjugation by a transformation of $\mathbb{T}^1$. Thus, all automorphisms of $\mathcal{E}$ are geometric, in the sense that they are induced by the action of $\mathcal{E}$ on $\mathbb{T}^1$.

2 Discontinuity Growth

For a map $f \in \mathcal{E}$, let $D(f)$ denote the set of points at which f is discontinuous as a map $\mathbb{T}^1 \to \mathbb{T}^1$. Let $D_{np}(f)$ be those discontinuities of f which are not periodic:

$$D_{np}(f) = D(f) \setminus \mathrm{Per}(f).$$

Note that if f is an infinite-order map and $D(f)$ is nonempty, then $D_{np}(f)$ is also nonempty. If $x \in D_{np}(f)$, both the forward and backward orbits of x eventually consist entirely of points at which f is continuous, since $D_{np}(f)$ is a finite set of points with nonperiodic orbits. Moreover, for each $x \in D_{np}(f)$, there is some $k \geq 0$, such that $f^{-k}(x)$ is the last point of $D_{np}(f)$ encountered in the negative orbit of x. In particular, f is continuous at all negative iterates $f^{-n}(x)$ for which $n > k$.

Definition 2.1. A nonperiodic discontinuity $x \in D_{np}(f)$ is a *fundamental discontinuity* if f is continuous at all negative iterates of x:

$$\left\{f^{-i}(x)\right\}_{i=1}^{\infty} \subseteq \mathbb{T}^1 \setminus D(f).$$

The set of fundamental discontinuities of f is denoted $D_F(f)$.

Thus, any point in $D_{np}(f)$ is either a fundamental discontinuity or a forward iterate of a fundamental discontinuity. In particular, the set of fundamental discontinuities is nonempty whenever $D(f)$ is nonempty and f has infinite order.

Let f_- denote the left-continuous form of f:

$$f_-(x) = \begin{cases} \lim_{y \to x^-} f(y), & \text{if } f \text{ is discontinuous at } x; \\ f(x), & \text{otherwise.} \end{cases}$$

Similarly, $f_+ = f$ may used to denote the original right-continuous map. Observe that $(f_-)^n = (f^n)_-$ and $(f_+)^n = (f^n)_+$ for all integers n; such compositions are thus denoted f_-^n and f_+^n without ambiguity. It follows that an iterate f^n is continuous at x if and only if

$$f_-^n(x) = f_+^n(x).$$

The sets

$$\left\{f_+^n(x)\right\}_{n=0}^{\infty} \quad \text{and} \quad \left\{f_-^n(x)\right\}_{n=0}^{\infty}$$

are called the *right* and *left (forward) orbits* of x, respectively.

Let $x \in D_{np}(f)$ be a fundamental discontinuity. By the definition of $D_{np}(f)$, the right orbit $\{f_+^n(x)\}$ is nonperiodic. Since x is fundamental, f is continuous at all points in the negative orbit of x. Thus, the left and right orbits coincide for negative iterates of f, and it follows that the left orbit of x is also nonperiodic. Therefore, since the set $D(f)$ is finite and the left and right forward orbits of x are nonperiodic, both of these forward orbits eventually consist entirely of points at which f is continuous.

Definition 2.2. The *stabilization time* of an interval exchange f is the smallest positive integer n_0, such that f is continuous at $f_+^n(x)$ and $f_-^n(x)$ for all $n \geq n_0$ and for all fundamental discontinuities x. For a fundamental discontinuity x, if $f_+^{n_0}(x) = f_-^{n_0}(x)$, then $f_+^n(x) = f_-^n(x)$ for all $n \geq n_0$, since f is continuous at all points in question. Such a fundamental discontinuity is said to be *eventually resolving*. Similarly, $f_+^{n_0}(x) \neq f_-^{n_0}(x)$ implies $f_+^n(x) \neq f_-^n(x)$ for all $n \geq n_0$; in this case, x is said to be *nonresolving*.

Figure 5 gives examples of interval exchanges exhibiting the two types of fundamental discontinuity. The map f has fundamental discontinuities at α and $\alpha + \beta$, and it may be checked that both of these discontinuities are nonresolving. The map g is a product of two restricted irrational rotations. It has fundamental discontinuities at $\frac{1}{2} - \gamma$ and $1 - \delta$, both of which are eventually resolving.

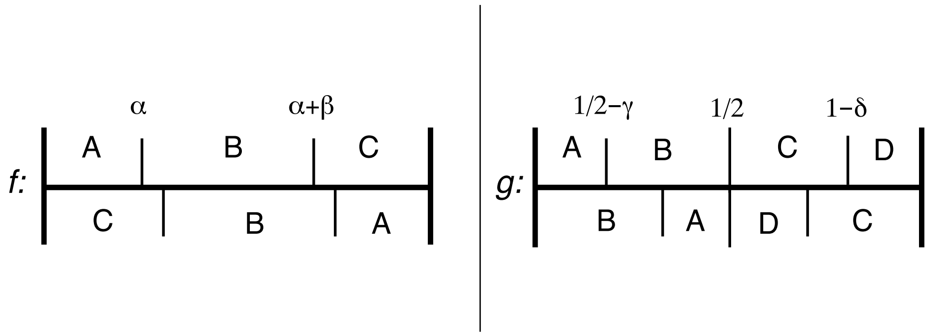

Figure 5: Types of fundamental discontinuities; all parameters are irrational

Fundamental discontinuities are so-named because they completely control the asymptotics of $d(f^n)$.

Proposition 2.3. *For any infinite-order interval exchange f, exactly one of the following holds:*

(a) *All fundamental discontinuities of f are eventually resolving, in which case $d(f^n)$ is bounded independently of n.*

(b) *The map f has $k \geq 1$ nonresolving fundamental discontinuities, in which case $d(f^n)$ has linear growth on the order of $k|n|$.*

Proof. If the map f is continuous, then conclusion (a) holds vacuously. When $D(f)$ is nonempty, the discontinuities of f^n are contained in the set

$$C_n = \bigcup_{i=0}^{n-1} f^{-i}\left(D(f)\right).$$

For $p \in C_n$, the left and right orbits (and hence the f^n-continuity status) of p are determined by the left and right orbits of the first f-discontinuity x that p meets in its forward f-orbit. Suppose $x = f^j(p)$, for $j \geq 0$. If $n > j$, then

$$f^n_{\pm}(p)) = f^{n-j}_{\pm}(f^j_{\pm}(p)) = f^{n-j}_{\pm}(x),$$

where $f^j_{\pm}(p) = x$ only because f is continuous at the points

$$\{p, f(p), \ldots, f^{(j-1)}(p)\}.$$

It will first be shown that non-fundamental discontinuities of f induce a uniformly bounded number of discontinuities for an iterate f^n. Let $x \in D_{np}(f)$ be a non-fundamental discontinuity. Then some negative iterate $y = f^{-j}(x)$, $j \geq 1$, is a fundamental discontinuity. Thus, for any $n \geq 1$, the discontinuity x only determines the f^n-continuity status of points in the set

$$\{x, f^{-1}(x), \ldots, f^{-(j-1)}(x)\}.$$

The status of all other preimages $f^{-k}(x)$, such that $k \geq j$, is determined by the fundamental discontinuity y. Consequently, the number of points whose f^n-continuity status is determined by the non-fundamental discontinuity x is bounded by j, independently of n. Similarly, there is a uniform bound to the number of points whose f^n-continuity status is determined by a periodic discontinuity of f.

Next, let x be an eventually resolving fundamental discontinuity of f, and let n_0 be the stabilization time of f. Then,

$$f_+^n(x) = f_-^n(x)$$

for all $n \geq n_0$. Suppose $n \geq n_0$ and k is such that $0 \leq k \leq (n - n_0)$. The right and left orbits of $f^{-k}(x)$ are determined by the right and left orbits of x:

$$f_+^n(f^{-k}x) = f_+^{n-k}(x) = f_-^{n-k}(x) = f_-^n(f^{-k}x),$$

where the middle equality holds because $n - k \geq n_0$. Thus, f^n is continuous at $f^{-k}(x)$, whenever $0 \leq k \leq (n - n_0)$. It follows that for all $n \geq n_0$, $\{x, f^{-1}(x), \ldots, f^{-(n-1)}(x)\}$ contains at most n_0 discontinuities of f^n. Therefore, $d(f^n)$ is bounded if all fundamental discontinuities of f are eventually resolving.

Alternately, suppose that x is a nonresolving fundamental discontinuity. Then

$$f_+^n(x) \neq f_-^n(x)$$

for all $n \geq n_0$. By an argument similar to the one above, it follows that f^n is discontinuous at $f^{-k}(x)$, for all k such that $0 \leq k \leq (n - n_0)$. Thus, if $n \geq n_0$, f^n has at least $n - n_0$ discontinuities in the set $\{x, f^{-1}(x), \ldots, f^{-(n-1)}(x)\}$. Since n_0 is fixed relative to n, this implies that $d(f^n)$ has linear growth.

Consequently, the presence of at least one nonresolving fundamental discontinuity implies linear growth of $d(f^n)$, and the presence of k nonresolving fundamental discontinuities implies $d(f^n) \sim kn$.

□

3 Classification of maps with bounded discontinuity growth

The most simple example of an infinite-order interval exchange with bounded discontinuity growth is an irrational rotation r_α, for which $d(r_\alpha^n)$ is always zero. The discontinuity growth rate of a map is invariant under conjugation, so we begin by stating a theorem of Li [**4**] which gives necessary and sufficient conditions for an interval exchange to be conjugate to an irrational rotation. For $f \in \mathcal{E}$, let $\delta(f)$ represent the number of intervals exchanged by f when viewed as a map $[0,1) \to [0,1)$:

$$\delta(f) = \min\{n : f = f_{(\pi,\lambda)} \text{ for some } \pi \in \Sigma_n, \lambda \in \Lambda_n\}.$$

Theorem (Li [**4**])**.** *An interval exchange map f is conjugate to an irrational rotation if and only if the following hold:*

(i) $\delta(f^n)$ *is bounded by some positive integer* N*,*

(ii) f^n *is minimal for all* $n \in \mathbb{N}$*, and*

(iii) *There are integers* $k > 0$ *and* $M \geq 2^{N^3+3N^2}$ *such that* $\widetilde{f} = f^k$ *satisfies* $\delta(\widetilde{f}) = \delta(\widetilde{f}^2) = \cdots = \delta(\widetilde{f}^M)$.

The quantities $\delta(f)$ and $d(f)$ are related, but they do not differ by a uniform constant for all $f \in \mathcal{E}$. For a rotation r_α, $\delta(r_\alpha) = 2$ and $d(r_\alpha) = 0$, while $\delta(f) = d(f) = 3$ for any map $f = f_{(\pi,\lambda)}$ with permutation $\pi = (3,2,1)$. It may be checked that the continuity status of the points 0 and $f^{-1}(0)$ account for any difference between $\delta(f)$ and $d(f)$; the function $\delta(f)$ always counts these points as left endpoints of a partition interval of f, but one or both of these points may fail to be a discontinuity of f when viewed as a map $\mathbb{T}^1 \to \mathbb{T}^1$. Consequently, some care must taken with condition (iii) in restating the above theorem in terms of the discontinuity number d. Conceivably, one might observe $d(f^k)$ to be constant over a large range of k while $\delta(f^k)$ is changing frequently.

This difficulty may be overcome by a good choice of the base point on $\mathbb{T}^1$. Presenting an interval exchange as defined on $[0,1)$ amounts to specifying a base point 0 at which to cut the circle. Choosing a new base point amounts to conjugation by a rotation; since the conclusion of Li's theorem is up to conjugacy, there is no loss in changing the base point. If f is replaced with a conjugate by a rotation, it may be assumed that f is continuous at all points of the orbit $\mathcal{O}_f(0)$. Consequently, $d(f^n) = \delta(f^n) - 2$ for all integers n, and observing $d(f^n)$ to be constant is now equivalent to observing that $\delta(f^n)$ is constant.

Theorem (Alternate Version of Li's Theorem)**.** *An interval exchange map f is conjugate to an irrational rotation if and only if the following hold:*

(i) *$d(f^n)$ is bounded by some integer N,*

(ii) *f^n is minimal for all $n \in \mathbb{N}$, and*

(iii) *after redefining the base point (conjugating by a rotation) so that f is continuous on the orbit of 0, there are integers $k > 0$ and $M \geq 2^{N^3+3N^2}$ such that $\widetilde{f} = f^k$ satisfies $d(\widetilde{f}) = d(\widetilde{f}^2) = \cdots = d(\widetilde{f}^M)$.*

Given this version of the theorem, it may now be seen to what extent the conditions *(ii)* and *(iii)* hold when it is only assumed that $d(f^n)$ is bounded. To introduce some terminology, a finite union J of half-open intervals is a *minimal component* of f if J is f-invariant and the f-orbit of any $x \in J$ is dense in J. It is shown in [**1**] and [**8**] that for any interval exchange f, the set of non-periodic points of f decomposes into finitely many minimal components.

Lemma 3.1. *Suppose that f is minimal and $d(f^n)$ is bounded. Then for some $k \in \mathbb{N}$, all nontrivial iterates f^{nk} are minimal when restricted to each minimal component of f^k.*

Proof. Suppose that no such integer k exists. Then f is minimal, but for some $k_1 = m_1 > 1$, f^{m_1} has multiple minimal components. Suppose that this integer k_1 has been chosen to be as small as possible. Since f and f^{m_1} commute, f permutes the minimal components of f^{m_1}. This permutation induced by f is transitive since f is minimal, and it must be of order m_1, by the choice of m_1. Thus f^{m_1} has exactly m_1 minimal components, denoted by $J_{1,1}, \ldots, J_{1,m_1}$.

It has been assumed that no power f^k is minimal for all iterates f^{kn} when restricted to any of its minimal components. Thus, there exists a smallest integer $k_2 > 1$ such that f^{m_2}, where $m_2 = k_1k_2$, is not minimal when restricted to some minimal component of f^{m_1}. Suppose this component is $J_{1,1}$. The map f^{m_1} permutes the minimal components of f^{m_2} which are contained in $J_{1,1}$; f^{m_1} acts minimally on $J_{1,1}$, and so this permutation must be transitive and have order k_2. Additionally, the original map f permutes the minimal components of f^{m_2}; since it also transitively permutes the minimal components of f^{m_1}, it follows that f^{m_2} must have k_2 minimal components in each one of the $J_{1,j}$. Thus f^{m_2} has exactly $k_2k_1 = m_2$ minimal components.

By the assumption that no k satisfies the conclusion of the lemma, this process may continue indefinitely. In particular, there are sequences of integers $k_i > 1$ and $m_i = \Pi_{j=1}^{i} k_j$, such that f^{m_i} has exactly m_i minimal components.

To arrive at a contradiction with the hypothesis that $d(f^n)$ is bounded, observe that if a map g has $m > 1$ minimal components $J_1, \dots, J_m$, then it must have at least m discontinuities. To see this, consider a left-boundary point x_i of J_i. Since some iterate of x_i will eventually fall in the interior of J_i, it follows that the orbit of each x_i must contain a discontinuity of g. Since these orbits are distinct, the map must have at least m discontinuities. Thus, it is impossible for f^n to have an arbitrarily large number of minimal components if $d(f^n)$ is bounded. □

Remark 3.2. It seems plausible that the above lemma should hold in general; i.e., the condition that $d(f^n)$ is perhaps not necessary. However, the argument above strongly uses this assumption and breaks down without it.

Lemma 3.3. *Suppose f has infinite order and $d(f^n)$ is bounded. Then for some $N \in \mathbb{N}$, $d(f^{nN})$ is constant over all $n \in \mathbb{N}$.*

Proof. By initially replacing f with an iterate, it may be assumed that $\mathrm{Per}(f) = \mathrm{Fix}(f)$. Let $D_F = \{x_1, \dots, x_k\}$ be the fundamental discontinuities of f. Since $d(f^n)$ is bounded, each x_i is eventually resolving. All other non-fixed discontinuities are found in the forward orbits of the fundamental discontinuities. Choose an integer $N_1 > 0$ such that any point of $D_{np}(f)$ may be reached from D_F by at most N_1 iterates of f. Such an N_1 exists since the set $D_{np}(f)$ is finite.

Choose N_2 such that the right and left orbits of all discontinuities in $D_{np}(f)$ are stabilized after N_2 iterates of f. In the situation where a non-fundamental discontinuity $x \in D_{np}(f)$ is fixed from the left (i.e., $f_-(x) = x$), it is the case that $f_+^n(x) \neq f_-^n(x)$ for all $n \geq 1$, since the right orbit of x is nonperiodic. Otherwise, both the right and left forward orbits of any $x \in D_{np}(f)$ eventually consist entirely of continuity points of f. Thus, the notion of stabilization time is well-defined for all $x \in D_{np}(f)$.

Finally, choose $N > N_1 + N_2$. It will be shown that $d(f^{kN})$ is constant over all $k \in \mathbb{N}$. Since $\mathrm{Per}(f) = \mathrm{Fix}(f)$, the set of fixed discontinuities is identical for all nonzero iterates of f. Thus, it suffices to only consider the set $D_{np}(f^N)$ of non-fixed discontinuities of f^N; any such point must be of the form $f^{-i}(x)$, where $x \in D_{np}(f)$ and $0 \leq i < N$. The non-fixed discontinuities of f are contained in the set

$$\bigcup_{i=0}^{N_1} f^i(D_F).$$

It follows that the non-fixed discontinuities of f^N are contained in the set

$$\bigcup_{i=-(N-1)}^{N_1} f^i(D_F).$$

Let

$$P = D(f^N) \cap \left(\bigcup_{i=1}^{N_1} f^i(D_F) \right), \quad Q = D(f^N) \cap \left(\bigcup_{i=-(N-1)}^{0} f^i(D_F) \right).$$

Consider a point $x \in P$. Since this is a discontinuity of f^N, the forward f-orbit of x must encounter a discontinuity of f whose right and left orbits control the continuity status of x. Since x is in P, this controlling discontinuity is non-fundamental, and it must be encountered within N_1 iterates of f. Since $N > N_1 + N_2$, the inequality between $f^N(x^+)$ and $f^N(x^-)$ occurs at a place where the right and left orbits of the controlling discontinuity have already stabilized. Thus, the right and left orbits of the controlling discontinuity are nonresolving, and it follows that x is a discontinuity of f^n, for all $n \geq N$. In particular, x is a discontinuity for all f^{kN}. Similarly, if a point in $\bigcup_{i=1}^{N_1} f^i(D_F)$ is a point of continuity for f^N, it must be a point of continuity for all f^{kN}.

Next, consider a point $x \in Q$. This point is a discontinuity of f^N whose f-orbit is controlled by a fundamental discontinuity x_i of f. Observe that under f^{kN}, the image of x is contained in

$$\bigcup_{i=(k-1)N+1}^{kN} f^i(D_F).$$

Consequently, if $k \geq 2$, the right and left orbits of x (which are controlled by the right and left orbits of the fundamental discontinuity x_i) have resolved once f^{kN} iterates have been applied to x. Thus x, as well as all other points in $\bigcup_{i=-(N-1)}^{0} f^i(D_F)$, are continuity points for f^{kN}, $k \geq 2$. In general, the f^{kN}-continuity status of any point in $\bigcup_{i=-(kN-1)}^{0} f^i(D_F)$ is controlled by the right and left orbits of a fundamental discontinuity. Since these orbits all resolve within N iterates, it follows that

$$D(f^{kN}) \cap \left(\bigcup_{i=-(kN-1)}^{0} f^i(D_F) \right) = f^{-(k-1)N}(Q).$$

The previous two paragraphs have shown that

$$D(f^{kN}) = P \cup f^{-(k-1)N}(Q).$$

This union is always disjoint, so the size of $D(f^{kN})$ is constant over all $k \in \mathbb{N}$. Since

$$d(f^{kN}) = \left|D(f^{kN})\right| + \left|\{\text{fixed discontinuities of } f^{kN}\}\right|,$$

and the second term in this sum is constant over all iterates of f, it follows that $d(f^{kN})$ is constant over all $k \in \mathbb{N}$, as desired. □

Proof of Theorem 1.2. Since f may be replaced with a power of itself, it may be assumed that $\mathrm{Per}(f) = \mathrm{Fix}(f)$. By applying Lemma 3.1 to the restriction of f on each of its minimal components, there is some k such that any f^{nk} is minimal when restricted to any minimal component $J_1, \ldots, J_m$ of f^k. Since the result is up to conjugacy in $\mathcal{E}$, it may be assumed that the minimal components J_i are all intervals.

Consider the restriction f_j of f^k to its minimal component J_j. It suffices to show that f_j is conjugate to an irrational rotation. The function $d(f_j^n)$ is bounded, and by construction f_j^n is minimal for all $n > 0$. If necessary, conjugate f_j by a rotation to assure that f_j is continuous at all points of the orbit of 0. By Lemma 3.3, there exists N_j such that $d(f_j^{nN_j})$ is constant for all n. Consequently, the alternate version of Li's theorem applies to the restricted map f_j, and so this map is conjugate to an irrational rotation. □

4 Proof of Theorem 1.3

We prove in this section that $\mathcal{E}$ does not contain distortion elements. To achieve this, it suffices to prove that an infinite-order interval exchange is not a distortion element in any finitely generated subgroup of $\mathcal{E}$ which contains it.

By Theorem 1.1, the iterates of f have linear or bounded discontinuity growth. Suppose first that f has linear discontinuity growth. Let $S = \{g_1, \ldots, g_k\}$ generate a subgroup of $G < \mathcal{E}$ which contains f, and let

$$M = \max_i \{d(g_i)\}.$$

Then

$$d(f^n) \leq M|f^n|_S,$$

since f^n may be expressed as a composition of $|f^n|_S$ elements from the set of generators. Consequently, linear growth of $d(f^n)$ implies linear growth of $|f^n|_S$, and thus f is not a distortion element of G.

Suppose now that f has infinite order and bounded discontinuity growth, and again suppose $f \in G = \langle g_1, \ldots, g_n \rangle < \mathcal{E}$. By Theorem 1.2, after conjugation and replacing f by an iterate it may be assumed that f is a product of disjointly supported infinite-order restricted rotations.

Let $r_{\alpha,\beta}$ denote one of these rotations, and assume first that $\alpha \notin \mathbb{Q}$. Let V be the $\mathbb{Q}$-vector subspace of $\mathbb{R}/\mathbb{Q}$ which is generated by the set of distances an element of G may translate a point of $\mathbb{T}^1$. The space V is a finite-dimensional $\mathbb{Q}$-vector space, since it is generated by the components of the translation vectors $\omega(g_i), 1 \leq i \leq n$.

Fix a basis for V which includes the class $[\alpha] \in \mathbb{R}/\mathbb{Q}$ and the class $[\beta]$ if $\beta \notin \mathbb{Q}$. Let

$$P_\alpha : V \to \mathbb{Q}$$

be the linear projection which returns the $[\alpha]$-coordinate of a vector with respect to this basis. For $p \in \mathbb{T}^1$, define the function $\phi_{\alpha,p} : G \to \mathbb{Q}$ by

$$\phi_{\alpha,p}(g) = P_\alpha(g(p) - p).$$

Note that the maps $\phi_{\alpha,p}$ satisfy the cocycle relation

$$\phi_{\alpha,p}(fg) = \phi_{\alpha,p}(g) + \phi_{\alpha,g(p)}(f).$$

The map f rotates by α mod β on the interval $[0, \beta)$ and $P_\alpha(\beta) = 0$, so it follows that

$$\phi_{\alpha,0}(f^n) = n, \text{ for all } n \in \mathbb{Z}.$$

Now consider the generators $g_1, \ldots, g_n$. Each one of these maps induces only finitely many distinct translations, namely the components of $\omega(g_i)$. Consequently, there is a constant $M > 0$ such that

$$|\phi_{\alpha,p}(g_i)| \leq M, \text{ for all } p \in \mathbb{T}^1, 1 \leq i \leq n.$$

Thus, for any $g \in G$,

$$|\phi_{\alpha,0}(g)| \leq M|g|_S.$$

In particular,

$$n = \phi_{\alpha,0}(f^n) \leq M|f^n|_S, \text{ for all } n \in \mathbb{Z},$$

which implies linear growth for $|f^n|_S$. Consequently, the map f is not a distortion element in G.

Suppose we are in the case where f is a product of infinite-order rotations, but all of these rotations are by some $\alpha_i \in \mathbb{Q}$ (mod $\beta_i \notin \mathbb{Q}$). The argument above fails in this case because the map $\phi_{\alpha,p}$ is not well-defined when α is rational. However, a similar argument can be made by tracking the contribution from the irrational number β. Choose a new basis for V which contains $[\beta]$, and consider the map $\phi_{\beta,0}$. The rotation by α mod β on $[0, \beta)$ contributes

$(-1)\beta$ for every loop the iterated rotation makes around this interval. Thus, there exists some constant $C > 0$, (for instance, any $C > \beta/\alpha$), such that

$$|\phi_{\beta,0}(f^n)| \geq \frac{n}{C},$$

It is still the case that there is a constant $M > 0$ such that

$$\phi_{\beta,0}(f^n) \leq M|f^n|_S,$$

which again implies linear growth for $|f^n|_S$. Thus, no infinite-order element of $\mathcal{E}$ is a distortion element.

5 Classification of Centralizers in $\mathcal{E}$

5.1 The bounded growth case

For $f \in \mathcal{E}$, let $C(f)$ denote the centralizer of f in the group $\mathcal{E}$:

$$C(f) = C_{\mathcal{E}}(f) := \{g \in \mathcal{E} : fg = gf\}.$$

If f is minimal, then the structure of $C(f)$ is primarily determined by the discontinuity growth of f. In considering the situation where $d(f^n)$ is bounded, the first case to consider is when $f = r_\alpha$ is an irrational rotation. Let $R = \{r_\alpha : \alpha \in \mathbb{R}/\mathbb{Z}\}$ denote the subgroup of rotation maps.

Lemma 5.1. *If α is irrational, then $C(r_\alpha)$ is the rotation group R.*

Proof. (See [**4**] for an alternate proof.) Suppose that $g \in \mathcal{E}$ commutes with r_α, in which case $g = r_\alpha^{-1} g r_\alpha$. Since r_α is continuous as a map $\mathbb{T}^1 \to \mathbb{T}^1$, this conjugacy implies that the discontinuity set $D(g)$ is r_α-invariant. Consequently, if $D(g)$ is a nonempty set, it must be infinite, which is impossible. Thus, $D(g)$ is empty, which implies $g \in R$, as rotations are the only continuous interval exchanges. $\square$

If f is minimal and $d(f^n)$ is bounded, by Theorem 1.2 some power f^k is conjugate to a product of disjointly supported infinite-order restricted rotations. Suppose that k is chosen to be as small as possible, and let J_i, $1 \leq i \leq l$, denote the minimal components of f^k. Replace f by a conjugate so that the J_i are intervals, and let r_i denote the restricted rotation supported on J_i induced by f^k. Since f is minimal and commutes with f^k, f transitively permutes the J_i and induces conjugacies between all of the r_i. Consequently, the J_i are

all intervals of length $1/l$, and each r_i rotates by the same proportion of $1/l$. Let R_i denote the rotation group supported on the interval J_i. Then f^k is an element of the diagonal subgroup of

$$R_1 \times \cdots \times R_l,$$

and it follows from Lemma 5.1 that

$$C(f^k) = (R_1 \times \cdots \times R_l) \rtimes \Sigma_l,$$

where Σ_l is the embedding of the symmetric group which permutes the J_i by translation.

Since $C(f)$ is a subgroup of the virtually abelian group $C(f^k) \cong (\mathbb{R}/\mathbb{Z})^l \rtimes \Sigma_l$, it follows that $C(f)$ is also virtually abelian. In addition, $f \in C(f^k)$ implies that f has the form

$$f = r_1 \cdots r_l \sigma,$$

where $r_i \in R_i$ and σ is a permutation of the J_i by translation. In particular, f commutes with the diagonal subgroup in $R_1 \times \cdots \times R_l$, and we have proved the following.

Proposition 5.2. *If f is minimal and $d(f^n)$ is bounded, then $C(f)$ is virtually abelian and contains a subgroup isomorphic to $\mathbb{R}/\mathbb{Z}$.*

5.2 The linear growth case

Next, suppose that f is minimal and $d(f^n)$ exhibits linear growth. The discontinuity structure of f and its iterates is significantly more complicated than the bounded case. Any map g which commutes with f must preserve this structure, and thus one would expect the centralizer of f to be significantly smaller than in the bounded discontinuity situation.

Proposition 5.3. *If f is minimal and $d(f^n)$ has linear growth, then $C(f)$ is virtually cyclic.*

To prove this, let $D = D(f)$ be the discontinuity set of f and let $D_{NR} = \{x_1, \ldots, x_k\}$ be the set of nonresolving fundamental discontinuities of f, which is nonempty by Proposition 2.3. Let n_0 be the *symmetric stabilization time* for f: n_0 is the minimal positive integer such that f is continuous at $f^i(x)$, for all i such that $|i| \geq n_0$ and all $x \in D$. The following lemma states that if a sufficiently long piece of f-orbit contains enough discontinuity points of a large power of f, then the f-orbit must contain a nonresolving fundamental discontinuity of f.

Lemma 5.4. *Suppose f is minimal and has symmetric stabilization time n_0. Let $M > 3n_0$, and suppose that for some $y \in \mathbb{T}^1$ the set*

$$B = \{y, f^{-1}(y), \dots, f^{-M+n_0}(y)\}$$

contains strictly more that $2n_0 + 1$ discontinuities of f^M. Then some $f^k(y)$, where $|k| \leq M$, is a nonresolving fundamental discontinuity of f.

Proof. Let y_m denote $f^m(y)$, and let $j \in \mathbb{N}$ be the smallest positive integer such that f^M is discontinuous at y_{-j}. Since f^M is discontinuous at y_{-j}, this point has a controlling f-discontinuity at $y_{\tilde{k}} \in D(f)$, where $-j \leq \tilde{k} \leq -j + M - 1$. Consequently, there must be a fundamental discontinuity of f at some y_k, where $-j - n_0 \leq k \leq -j + M - 1$.

Since B contains more than $2n_0$ f^M-discontinuities at points y_{-i} with $i > j$, there are more than n_0 f^M-discontinuities whose status is controlled by y_k. In particular, at least one of the f^M-discontinuities in B is induced by the stabilized behavior of y_k, which implies that y_k is a nonresolving fundamental discontinuity of f. $\square$

Proof of Proposition 5.3. Suppose that $gf = fg$. Let $N \in \mathbb{N}$ be such that

$$N \gg n_0 \text{ and } N \gg d(g).$$

Let $x \in D_{NR}$. Since x is a nonresolving, the set

$$A = \{x, f^{-1}x, f^{-2}x, \dots, f^{-(N-n_0)}x\}$$

consists entirely of discontinuity points of f^N.

Since f and g commute, $g^{-1}f^N g = f^N$ is discontinuous at all points of A. Consider how this composition acts upon the set A :

$$\begin{array}{c}
A = \{x, f^{-1}x, \dots, f^{-(N-n_0)}x\} \\
\downarrow g \\
g(A) = \{gx, f^{-1}(gx), \dots, f^{-(N-n_0)}(gx)\} \\
\downarrow f^N \\
f^N g(A) = \{f^N(gx), f^{N-1}(gx), \dots, f^{n_0}(gx)\} \\
\downarrow g^{-1} \\
f^N(A) = \{f^N x, f^{N-1}x, \dots, f^{n_0}x\}
\end{array}$$

The cardinality of A is significantly larger than $d(g)$, which implies that g acts continuously on most points of A in the first stage of the above composition. Similarly, g^{-1} acts continuously on most points of $f^N g(A)$ in the third

stage. However, since $g^{-1}f^N g$ is discontinuous at all points of A, it follows that f^N is discontinuous at most of the points in

$$\{gx, f^{-1}(gx), \ldots, f^{-(N-n_0)}(gx)\}.$$

By Lemma 5.4, it follows that some f-iterate of $g(x)$ must be in D_{NR}.

The preceding paragraphs show that $g \in C(f)$ permutes the f-orbits of the points in $D_{NR} = \{x_1, \ldots, x_k\}$. In particular, there is some integer i and some $x_j \in D_{NR}$ such that

$$g(x_1) = f^i(x_j).$$

This relation determines g on the entire f-orbit of x_1:

$$g(f^n x_1) = f^n(gx_1) = f^{n+i}x_j.$$

Since the orbit $\mathcal{O}_f(x_1)$ is dense, this relation fully determines g.

For each j such that $1 \leq j \leq k$, let h_j denote the unique interval exchange in $C(f)$ such that

$$h_j(x_1) = x_j,$$

if such a map exists. Then, if $g \in C(f)$ satisfies $g(x_1) = f^k(x_j)$, it follows that $g = f^k h_j$. In particular, $\{h_i\}$ is a set of representatives for the finite quotient group $C(f)/\langle f\rangle$, and consequently $C(f)$ is virtually cyclic. $\square$

5.3 Centralizers of finite order maps

For $n \geq 2$, the rotation $r_{1/n}$ induces a cyclic permutation of the intervals

$$I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right), \quad 1 \leq i \leq n.$$

Recall that the support of an interval exchange f is the complement of its set of fixed points. For $1 \leq j \leq n$, let $\mathcal{E}_{I_j}$ denote the subgroup of all interval exchanges whose support is contained in I_j. Note that the orientation-preserving affine bijection $I_j \to [0,1)$ induces an isomorphism $\mathcal{E}_{I_j} \cong \mathcal{E}$.

Consider the following subgroups in the centralizer $C(r_{1/n})$. Let $\mathcal{E}^n_\Delta$ represent the maps in $C(r_{1/n})$ which preserve the intervals I_i:

$$\mathcal{E}^n_\Delta = \{g \in C(r_{1/n}) : g(I_i) = I_i, \text{ for } 1 \leq i \leq n\}.$$

Note that $\mathcal{E}^n_\Delta$ is the diagonal subgroup of the product

$$\mathcal{E}_{I_1} \times \cdots \times \mathcal{E}_{I_n},$$

as induced by the natural isomorphisms $\mathcal{E} \cong \mathcal{E}_{I_i}$. In short, a map in $\mathcal{E}^n_\Delta$ acts on each of the I_j in the same manner, and so $\mathcal{E}^n_\Delta \cong \mathcal{E}_{I_j} \cong \mathcal{E}$.

Next, let P_n denote the subgroup of maps in $C(r_{1/n})$ which are invariant on $r_{1/n}$-orbits $\{x + k/n : k = 0, 1, \ldots, n-1\}$:

$$P_n = \left\{ g \in C(r_{1/n}) : \forall x \in \mathbb{T}^1, \exists k \in \mathbb{Z}, \text{ such that } g(x) = x + \frac{k}{n} \pmod 1 \right\}.$$

Fix $g \in P_n$, and consider a point $x = x_1 \in I_1$. Let

$$x_k = r^{k-1}_{1/n}(x_1) = x_1 + \frac{k-1}{n}, \; 2 \leq k \leq n,$$

denote the other points in the $r_{1/n}$-orbit of x, and let $\sigma_{g,x} \in \Sigma_n$ denote the permutation that g induces on $\{x_i\}$:

$$g(x_i) = x_{\sigma_{g,x}(i)}.$$

The permutation $\sigma_{g,x}$ commutes with the permutation $r : i \mapsto i + 1 \pmod n$, which implies that $\sigma_{g,x}$ must be a power of r. Thus, the transformation g is described by a right-continuous (and hence piecewise constant) map

$$\sigma_g : I_1 \to \langle r \rangle \cong \mathbb{Z}/n\mathbb{Z}.$$

Conversely, any such right-continuous map $I_1 \to \mathbb{Z}/n\mathbb{Z}$ with only finitely many discontinuities defines a map in P_n. Thus, P_n is isomorphic to the abelian group of right-continuous functions $I_1 \to \mathbb{Z}/n\mathbb{Z}$ having finitely many discontinuities.

Proposition 5.5. $C(r_{1/n}) = P_n \rtimes \mathcal{E}^n_\Delta$.

Proof. First, suppose $g \in P_n \cap \mathcal{E}^n_\Delta$. Then g preserves the intervals I_i, which implies that $\sigma_{g,x} = id$ for all $x \in I_1$. Thus $g = id$, and the subgroups P_n and $\mathcal{E}^n_\Delta$ have trivial intersection.

Next, suppose g is an arbitrary element of $C(r_{1/n})$. Construct $h \in P_n$ as follows. For $x = x_1 \in I_1$, define $\{x_i\}$ as before and let $\sigma_{h,x}$ be the permutation such that

$$g(x_i) \in I_{\sigma_{h,x}(i)}.$$

Observe that $\sigma_{h,x} \in \Sigma_n$ is well-defined since g maps an $r_{1/n}$-orbit $\{x_i\}$ to another $r_{1/n}$-orbit. Since g commutes with $r_{1/n}$, the permutation $\sigma_{h,x}$ is a power of the permutation r. Moreover, the function $x \mapsto \sigma_{h,x} \in \mathbb{Z}/n\mathbb{Z}[r]$ is right-continuous and has finitely many discontinuities, so it induces a map $h \in P_n$. From its construction, gh^{-1} preserves each interval I_i, and so $gh^{-1} \in \mathcal{E}^n_\Delta$. Thus $C(r_{1/n}) = P_n \cdot \mathcal{E}^n_\Delta$.

It remains to show that P_n is a normal subgroup of $C(r_{1/n})$. Let $g \in P_n$ and let $h \in \mathcal{E}^n_\Delta$. If $\{x_i\}$ is an $r_{1/n}$-orbit, then h maps it to some other $r_{1/n}$-orbit $\{y_i\}$, g permutes the orbit $\{y_i\}$, and h^{-1} maps $\{y_i\}$ back to $\{x_i\}$. Thus $h^{-1}gh$ is invariant on $r_{1/n}$-orbits, which implies that $h^{-1}gh \in P_n$. Consequently, $P_n \trianglelefteq C(r_{1/n})$. $\square$

Corollary 5.6. *For $n \geq 2$, let $G_n = P_n \rtimes \mathcal{E}^n_\Delta$ denote the centralizer of the rotation $r_{1/n}$, and let $G_1 = \mathcal{E}$. If f is any finite-order map, then $C(f)$ is isomorphic to a finite direct product of the G_i.*

Proof. Decompose $\mathbb{T}^1$ into finitely many nonempty

$$I_j = \mathrm{Per}_j(f) = \{x \in \mathbb{T}^1 : |\mathcal{O}(x)| = j\}.$$

This decomposition is finite because an interval exchange cannot have periodic points of arbitrarily large minimal period. After replacing f by a conjugate, it may be assumed that the I_j are intervals on which f acts by a finite-order rotation. The I_j are invariant under all $g \in C(f)$, and $C(f) \cap \mathcal{E}_{I_j}$ is isomorphic to G_j. $\square$

5.4 The general situation

Let f be any interval exchange. Let $J_1, \ldots, J_k$ denote the minimal components of f, let $A = \mathrm{Per}(f) \setminus \mathrm{Fix}(f)$ and let $B = \mathrm{Fix}(f)$. After replacing f by a conjugate, it may be assumed that each of these sets is an interval. Let f_i be the restriction of f to J_i, here defined on all of $\mathbb{T}^1$ by

$$f_i(x) = \begin{cases} f(x), & \text{if } x \in J_i \\ x, & \text{otherwise.} \end{cases}$$

Let $g \in C(f)$. The sets A and B are both g-invariant, but g may permute the minimal components J_i. However, if g maps J_i onto J_j, then g induces a conjugacy between f_i and f_j. After replacing f_j by a conjugate in $\mathcal{E}_{I_j}$, it may be assumed that

$$f_i = \tau_{ij} f_j \tau_{ij}$$

where τ_{ij} is the order-two map which interchanges J_i and J_j by translation and fixes all other points. Replace f by a further conjugate so that $f_i = \tau_{ij} f_j \tau_{ij}$ holds for all pairs $i \neq j$ such that f_i and f_j are conjugate, and let F be the group generated by all such τ_{ij}. Note that F is isomorphic to a direct product of symmetric groups, since the relation $i \sim j \Leftrightarrow (f_i$ is conjugate to $f_j)$ is an equivalence relation on $\{1, \ldots, k\}$. Let $C_i = C_{\mathcal{E}_{J_i}}(f) = C(f) \cap \mathcal{E}_{J_i}$ denote the subgroup of maps in $C(f)$ with support in J_i, and let $C_A = C(f) \cap \mathcal{E}_A$.

Theorem 5.7. *For any $f \in \mathcal{E}$, let $A = Per(f) \setminus Fix(f)$ and let $B = Fix(f)$. Then,*

$$C(f) \cong \left(\left(\prod_{i=1}^{k} C_i \right) \rtimes F \right) \times C_A \times \mathcal{E}_B,$$

where F is a direct product of symmetric groups, where each C_i is either an infinite virtually cyclic group or a subgroup of $(\mathbb{R}/\mathbb{Z})^n \rtimes \Sigma_n$ containing the diagonal in $(\mathbb{R}/\mathbb{Z})^n$, and where C_A is a direct product of finitely many factors $G_n = P_n \rtimes \mathcal{E}^n_\Delta$. The factors C_A and $\mathcal{E}_B$ are trivial if $Per(f) = \emptyset$, and the factors C_i and F are trivial if f has finite order.

Proof. It is clear that

$$C(f) \cong C_{\cup J_i}(f) \times C_A \times \mathcal{E}_B,$$

since these are disjoint and non-conjugate f-invariant sets which cover $\mathbb{T}^1$. The verification that

$$C_{\cup J_i}(f) \cong \left(\prod_{i=1}^{k} C_i \right) \rtimes F$$

is similar to the proof of Proposition 5.5. □

A corollary to this result states that the existence of periodic points for an interval exchange f is characterized by its centralizer.

Corollary 5.8. *For any $f \in \mathcal{E}$, $Per(f)$ is nonempty if and only if $C(f)$ contains a subgroup isomorphic to $\mathcal{E}$.*

Proof. The factors C_A and $\mathcal{E}_B$ both contain subgroups isomorphic to $\mathcal{E}$ if they are nontrivial, and at least one of these factors is nontrivial when $\mathrm{Per}(f)$ is nonempty.

It remains to show that if $\mathrm{Per}(f)$ is empty, then no subgroup of $C(f)$ is isomorphic to $\mathcal{E}$. In this case,

$$C(f) \cong \prod_{i=1}^{k} C_i \rtimes F,$$

where each C_i is either virtually cyclic or isomorphic to a subgroup of $(\mathbb{R}/\mathbb{Z})^n \rtimes \Sigma_n$ containing the diagonal. It may be seen that for any two infinite-order $g, h \in C(f)$, there are nontrivial powers g^j and h^k of these maps which commute. This property does not hold for the group $\mathcal{E}$. For instance, consider an irrational rotation r_α and any infinite-order map $f \in \mathcal{E}$ which is not a rotation; by Lemma 5.1, nontrivial powers of r_α and f do not commute. Thus, it is not possible to embed $\mathcal{E}$ as a subgroup of $C(f)$ when f has no periodic points. □

Corollary 5.9. *For any $f \in \mathcal{E}$ such that $f \neq id$, the index $[\mathcal{E} : C(f)]$ is uncountabe.*

Proof. From the structure of $C(f)$ given in the proposition, it suffices to consider the cases where f has finite order, where f is minimal with $d(f^n)$ bounded, and where f is minimal with linear discontinuity growth.

If f has finite order, it suffices to consider the case $f = r_{1/n}$. Fix an irrational α in $(0, 1)$, and note that the product of restricted rotations $r^{-1}_{\alpha\epsilon,\epsilon} r_{\alpha\epsilon',\epsilon'}$ is never an element of $C(f)$ for any $0 < \epsilon < \epsilon' < \frac{1}{n}$. Consequently, the $r_{\alpha\epsilon,\epsilon}$ provide an uncountable set of distinct coset representatives for $\mathcal{E}/C(f)$.

If f is minimal and $d(f^n)$ is bounded, consider a conjugate g of f^k which is a product of infinite-order restricted rotations on intervals of length $1/l$. Again, notice that for $0 < \epsilon < \epsilon' < 1/l$, the product $r^{-1}_{\alpha\epsilon,\epsilon} r_{\alpha\epsilon',\epsilon'}$ is not an element of $C(g)$. Consequently, the $r_{\alpha\epsilon,\epsilon}$ also provide an uncountable set of coset representatives for $\mathcal{E}/C(g)$, and it follows that $C(f^k)$ and $C(f)$ have uncountable index in $\mathcal{E}$.

If f is minimal with linear discontinuity growth, then by proposition 5.3, $C(f)$ is virtually cyclic. In particular, $C(f)$ is countable, which implies that $C(f)$ has uncountable index in $\mathcal{E}$. $\square$

6 Computation of Aut($\mathcal{E}$)

The proof of Theorem 1.6 is based on observing that an arbitrary $\Psi \in \mathrm{Aut}(\mathcal{E})$ preserves the structure of centralizers, which implies that Ψ preserves various dynamical properties of individual maps and subgroups in $\mathcal{E}$.

Lemma 6.1. *An interval exchange f is conjugate to an irrational rotation r_α if and only if the following conditions hold:*

(1) $C(f) \cong \mathbb{R}/\mathbb{Z}$*;*

(2) if $g \in C(f)$ *has infinite order, then* $C(g) = C(f)$*.*

Proof. By Lemma 5.1, conditions (1) and (2) hold if $f = r_\alpha$ is an irrational rotation, and these conditions are both preserved under conjugation.

Conversely, assume that f satisfies (1) and (2). By Corollary 5.8, $\mathrm{Per}(f)$ is empty. Next, suppose that some f^n has at least two minimal components, and denote them by J_i. Let g be the map which is equal to f on J_1 and fixes all other points. Then g has infinite order and commutes with f, so $C(g) \cong \mathbb{R}/\mathbb{Z}$ by condition (2). However, g has fixed points, and so $C(g)$ contains a subgroup

isomorphic to $\mathcal{E}$, which is impossible by Corollary 5.8. Thus, f^n is minimal for all $n \geq 1$.

Furthermore, f has bounded discontinuity growth. If not, then $C(f)$ is virtually cyclic by Proposition 5.3, which is not the case for $\mathbb{R}/\mathbb{Z}$. Consequently, by Theorem 1.2 some power f^k is conjugate to an irrational rotation. Since $C(f) = C(f^k)$, it follows that f is also conjugate to an irrational rotation. $\square$

Let $R < \mathcal{E}$ denote the group of circle rotations $\{r_\alpha : \alpha \in \mathbb{R}/\mathbb{Z}\}$. For any $f \in \mathcal{E}$, let Φ_f denote conjugation by f^{-1}; i.e., $\Phi_f(g) = fgf^{-1}$.

Corollary 6.2. *For any $\Psi \in Aut(\mathcal{E})$, Ψ maps the rotation group R to a conjugate. That is, there exists $g \in \mathcal{E}$ such that $\Psi(R) = gRg^{-1}$.*

Proof. Since conditions (1) and (2) in Lemma 6.1 are purely group theoretic, they are preserved by any automorphism Ψ. Fix an irrational rotation r_α. By the Lemma, $\Psi(r_\alpha)$ is conjugate to an irrational rotation. In particular, there is some $g \in \mathcal{E}$ and some irrational $\beta \in \mathbb{R}/\mathbb{Z}$ such that

$$\Psi(r_\alpha) = \Phi_g(r_\beta).$$

Then

$$\Psi(R) = \Psi(C(r_\alpha)) = C(\Psi(r_\alpha)) = C(\Phi_g(r_\beta)) = gRg^{-1}.$$

$\square$

A similar result holds for maps that are conjugate to an infinite-order restricted rotation $r_{\alpha,\beta}$.

Lemma 6.3. *An interval exchange f is conjugate to an infinite-order restricted rotation $r_{\alpha,\beta}$ if and only if the following hold:*

(1) $C(f) = \mathcal{E}_* \times H$, where $\mathcal{E}_* \cong \mathcal{E}$, $H \cong \mathbb{R}/\mathbb{Z}$, and $f \in H$;

(2) if $g \in H$ has infinite order, then $C(g) = C(f)$;

(3) for $h \in C(f)$, if the index $[C(f) : C(h) \cap C(f)]$ is finite and $C(h) \supsetneq C(h) \cap C(f)$, then h is a finite-order element of H.

Proof. Suppose that $f = r_{\alpha,\beta}$ with $\beta < 1$ and α/β irrational. Let $I = [\beta, 1)$. Then

$$C(r_{\alpha,\beta}) = \mathcal{E}_I \times R_\beta,$$

where $R_\beta \cong \mathbb{R}/\mathbb{Z}$ is the group of all restricted rotations $r_{\gamma,\beta}$ on $[0, \beta)$. Any other infinite-order element of R_β has the same centralizer as $r_{\alpha,\beta}$, and it follows that $r_{\alpha,\beta}$ satisfies conditions (1) and (2).

To verify condition (3) for $r_{\alpha,\beta}$, take $h \in C(r_{\alpha,\beta})$ and write $h = h_I r_{\gamma,\beta}$, where $h_I \in \mathcal{E}_I$ and $r_{\gamma,\beta} \in R_\beta$. Assume that $C(h)$ satisfies the hypotheses of condition (3). Note that

$$C(h) \cap C(r_{\alpha,\beta}) = C_{\mathcal{E}_I}(h_I) \times R_\beta,$$

and consequently,

$$[C(r_{\alpha,\beta}) : C(h) \cap C(r_{\alpha,\beta})] = [\mathcal{E}_I : C_{\mathcal{E}_I}(h_I)].$$

Corollary 5.9 states that the index $[\mathcal{E}_I : C_{\mathcal{E}_I}(h_I)]$ is infinite if h_I is not the identity. However, it has been assumed that this index is finite; thus $h_I = id$ and $h = r_{\gamma,\beta}$ is a restricted rotation. It has also been assumed that

$$C(h) \supsetneqq C(h) \cap C(r_{\alpha,\beta}) = C(r_{\alpha,\beta}),$$

and this is possible only if the rotation $h = r_{\gamma,\beta}$ has finite order.

Finally, observe that conditions (1)-(3) are all preserved under conjugation in $\mathcal{E}$. Consequently, they hold for any conjugate of $r_{\alpha,\beta}$.

Conversely, suppose that f is an interval exchange satisfying (1)-(3). Since $C(f)$ contains a subgroup isomorphic to $\mathcal{E}$, $A = \mathrm{Per}(f)$ is nonempty by Corollary 5.8. The map f does not have periodic points of arbitrarily large period, so $\mathrm{Fix}(f^k) = \mathrm{Per}(f^k) = A$ for some $k \geq 1$. Since f^k fixes A, $\mathcal{E}_A < C(f^k)$. By condition (2), $C(f^k) = C(f)$, and it follows that f fixes all points in A. Similarly, all infinite-order $g \in H$ must fix the set A, and consequently all maps in H must fix A. Thus, H is contained in $\mathcal{E}_B$, where $B = \mathbb{T}^1 \setminus A$.

Suppose now that f has $k \geq 2$ minimal components J_i, and let h be the map which equals f on the component J_1 and fixes all other points. Then h has infinite order and commutes with f. Thus, by Theorem 5.7,

$$\begin{aligned} C(f) &= \left(\left(\prod_{i=1}^{k} C_i \right) \rtimes F \right) \times \mathcal{E}_A, \text{ and} \\ C(h) &= C_1 \times \mathcal{E}_{A \cup J_2 \cup \cdots \cup J_k}, \end{aligned}$$

where $C_i = C(f) \cap \mathcal{E}_{J_i}$ and F is a finite group which permutes the J_i. In particular, $C(h) \cap C(f)$ contains $(\prod C_i) \times \mathcal{E}_A$, which has finite index in $C(f)$. In addition, $C(h)$ strictly contains $C(h) \cap C(f)$ since h has a larger fixed point set than f. Thus, condition (3) implies that h must have finite order, which is a contradiction. A similar argument may be applied to any infinite-order $g \in H$; thus, all such maps have a single minimal component, namely B.

Consider the natural isomorphism

$$\mathcal{E} \to \mathcal{E}_B.$$

Let $\widetilde{f}$ denote the preimage of $f \in \mathcal{E}_B$, and let $\widetilde{H}$ denote the preimage of H. Then all infinite-order $\widetilde{g} \in \widetilde{H}$ are minimal, and

$$C(\widetilde{g}) = C(\widetilde{f}) > \widetilde{H},$$

which implies that all infinite-order $\widetilde{g}$ have bounded discontinuity growth. As in the proof of Lemma 6.1, it follows that all $\widetilde{g} \in \widetilde{H}$ are simultaneously conjugate to irrational rotations. Back in the group $\mathcal{E}_B$, this implies that H is conjugate to a group of restricted rotations. □

As in the earlier case, observe that the three conditions in the previous proposition are purely group-theoretic. Consequently, they are all preserved by any automorphism Ψ, which implies the following corollary.

Corollary 6.4. *For any $\Psi \in Aut(\mathcal{E})$ and any f which is conjugate to a restricted rotation, $\Psi(f)$ is also conjugate to a restricted rotation.*

Let $\mathcal{P}$ denote the set algebra consisting of all finite unions of half-open intervals $[a, b) \subseteq \mathbb{T}^1$.

Proposition 6.5. *For any $\Psi \in Aut(\mathcal{E})$ and any nonempty $A \in \mathcal{P}$, there is a unique $B \in \mathcal{P}$ such that $\Psi(\mathcal{E}_A) = \mathcal{E}_B$.*

Proof. It suffices to consider $A \in \mathcal{P}$ to be a proper, nonempty subset of $\mathbb{T}^1$. Let $g \in \mathcal{E}$ be a map that is conjugate to an infinite-order restricted rotation, such that $\mathrm{Fix}(g) = A$. By Corollary 6.4, $\Psi(g)$ is also conjugate to a restricted rotation; let $B = \mathrm{Fix}(\Psi(g))$.

Observe that two infinite-order restricted rotations g and h commute if and only if one of the following holds:

(a) their supports coincide and they are simultaneously conjugate to elements in some R_β; or

(b) their supports are disjoint.

These conditions can be characterized in terms of centralizers: (a) implies that $C(g) = C(h)$, while (b) implies $C(g) \neq C(h)$. In particular, each condition is preserved by any automorphism of $\mathcal{E}$.

Any restricted rotation with support contained in $A = \mathrm{Fix}(g)$ commutes with g and has support disjoint from that of g. Consequently, all restricted

rotations in $\mathcal{E}_A$ must map under Ψ to restricted rotations with support in $B = \mathrm{Fix}(\Psi(g))$. The restricted rotations in $\mathcal{E}_A$ generate this subgroup; see [**8**] for a proof of this fact. Therefore, the image $\Psi(\mathcal{E}_A)$ is contained in $\mathcal{E}_B$.

Similarly, under Ψ^{-1} all restricted rotations with support in B are mapped to restricted rotations which commute with g and have support disjoint from that of g. Therefore, $\Psi^{-1}(\mathcal{E}_B) \subseteq \mathcal{E}_A$, and it follows that $\Psi(\mathcal{E}_A) = \mathcal{E}_B$. $\square$

6.1 Definition and properties of $\widetilde{\Psi}$

Given an automorphism $\Psi \in \mathrm{Aut}(\mathcal{E})$, Proposition 6.5 implies that there is a well-defined transformation

$$\widetilde{\Psi} : \mathcal{P} \to \mathcal{P},$$

defined by the relation

$$\Psi(\mathcal{E}_A) = \mathcal{E}_{\widetilde{\Psi}(A)}, \quad A \in \mathcal{P}.$$

In particular, $\widetilde{\Psi}(\mathbb{T}^1) = \mathbb{T}^1$ and $\widetilde{\Psi}(\emptyset) = \emptyset$, for all $\Psi \in \mathrm{Aut}(\mathcal{E})$. An element $f \in \mathcal{E}$ also induces a transformation $\widetilde{f} : \mathcal{P} \to \mathcal{P}$, defined by $\widetilde{f}(A) = f(A)$.

Proposition 6.6. *For all $\Psi \in Aut(\mathcal{E})$, the transformation $\widetilde{\Psi} : \mathcal{P} \to \mathcal{P}$ has the following properties:*

(1) $\widetilde{\Psi}$ is an automorphism of the set algebra $\mathcal{P}$.

(2) For any $f \in \mathcal{E}$, $\widetilde{\Psi(f)} = \widetilde{\Psi}\widetilde{f}\widetilde{\Psi}^{-1}$.

(3) The Lebesgue measure $\mu : \mathcal{P} \to [0,1]$ is $\widetilde{\Psi}$-invariant: $\mu(\widetilde{\Psi}(A)) = \mu(A)$.

Proof. To show that $\widetilde{\Psi}$ is a set algebra automorphism, it suffices to show $\widetilde{\Psi}$ preserves complements, inclusion, and unions in $\mathcal{P}$. If A and B are complements in $\mathcal{P}$, then the centralizer in $\mathcal{E}$ of $\mathcal{E}_A$ is $\mathcal{E}_B$, and vice versa. This same relation holds for $\Psi(\mathcal{E}_A) = \mathcal{E}_{\widetilde{\Psi}(A)}$ and $\Psi(\mathcal{E}_B) = \mathcal{E}_{\widetilde{\Psi}(B)}$, which implies $\widetilde{\Psi}(A)$ and $\widetilde{\Psi}(B)$ are complements.

For inclusion, notice that

$$A \subseteq B \Rightarrow \mathcal{E}_A \leq \mathcal{E}_B \Rightarrow \Psi(\mathcal{E}_A) \leq \Psi(\mathcal{E}_B) \Rightarrow$$

$$\mathcal{E}_{\widetilde{\Psi}(A)} \leq \mathcal{E}_{\widetilde{\Psi}(B)} \Rightarrow \widetilde{\Psi}(A) \subseteq \widetilde{\Psi}(B).$$

To verify that $\widetilde{\Psi}$ preserves unions, let A and B be elements of $\mathcal{P}$, and note that $\widetilde{\Psi}(A) \subseteq \widetilde{\Psi}(A \cup B)$ and $\widetilde{\Psi}(B) \subseteq \widetilde{\Psi}(A \cup B)$. Conversely, suppose that $\widetilde{\Psi}(A \cup B) \not\subseteq \widetilde{\Psi}(A) \cup \widetilde{\Psi}(B)$. To derive a contradiction, let

$$C = \widetilde{\Psi}(A \cup B) \setminus \left(\widetilde{\Psi}(A) \cup \widetilde{\Psi}(B)\right).$$

Then $C \in \mathcal{P}$ is nonempty, and there exists a non-identity interval exchange $f \in \mathcal{E}_C \leq \mathcal{E}_{\widetilde{\Psi}(A \cup B)}$. The map f centralizes both $\mathcal{E}_{\widetilde{\Psi}(A)}$ and $\mathcal{E}_{\widetilde{\Psi}(B)}$, so the map $\Psi^{-1}(f)$ centralizes $\mathcal{E}_A$ and $\mathcal{E}_B$. This implies $\Psi^{-1}(f)$ has support disjoint from both A and B. However, this is impossible since $\Psi^{-1}(f)$ is in $\mathcal{E}_{A \cup B}$. Thus, $\widetilde{\Psi}(A \cup B) \subseteq \widetilde{\Psi}(A) \cup \widetilde{\Psi}(B)$, which completes the verification that $\widetilde{\Psi}$ is a set algebra automorphism.

To prove property (2), recall $\Phi_f \in \mathrm{Aut}(\mathcal{E})$ denotes conjugation by f^{-1}. In particular, if $\widetilde{f}$ maps the set A to the set B, then Φ_f induces an isomorphism from $\mathcal{E}_A$ to $\mathcal{E}_B$. For any $g \in \mathcal{E}$,

$$\Psi \Phi_f \Psi^{-1}(g) = \Psi(f(\Psi^{-1}g)f^{-1}) = \Psi(f) \circ g \circ \Psi(f)^{-1}.$$

Thus $\Psi \Phi_f \Psi^{-1} = \Phi_{(\Psi f)}$, and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{E}_A & \xrightarrow{\Phi_f} & \mathcal{E}_B \\ {\scriptstyle \Psi}\downarrow & & \downarrow{\scriptstyle \Psi} \\ \mathcal{E}_{\widetilde{\Psi}(A)} & \xrightarrow[\Phi_{(\Psi f)}]{} & \mathcal{E}_{\widetilde{\Psi}(B)} \end{array}$$

Consequently, $\widetilde{\Psi(f)} = \widetilde{\Psi}\widetilde{f}\widetilde{\Psi}^{-1}$.

To prove that μ is invariant under $\widetilde{\Psi}$, it will first be shown that if $A, B \in \mathcal{P}$ are disjoint and $\mu(A) = \mu(B)$, then $\widetilde{\Psi}(A)$ and $\widetilde{\Psi}(B)$ are also disjoint and have equal measure. For such A and B, let $f \in \mathcal{E}$ be any interval exchange such that $\widetilde{f}(A) = B$. Then f induces a conjugacy between the subgroups $\mathcal{E}_A$ and $\mathcal{E}_B$, and $\Psi(f)$ induces a conjugacy between $\mathcal{E}_{\widetilde{\Psi}(A)}$ and $\mathcal{E}_{\widetilde{\Psi}(B)}$. By (2), $\widetilde{\Psi(f)}$ maps $\widetilde{\Psi}(A)$ to $\widetilde{\Psi}(B)$, and as a result, $\mu(\widetilde{\Psi}(A)) = \mu(\widetilde{\Psi}(B))$, which proves the initial claim.

To prove that $\mu(\widetilde{\Psi}(A)) = \mu(A)$ for any $A \in \mathcal{P}$, assume first that $\mu(A)$ is rational. Since any $\widetilde{\Psi}$ preserves finite disjoint unions, it may be further assumed that $\mu(A) = 1/n$. Lebesgue measure is invariant under any conjugacy Φ_f, so it finally suffices to consider the case $A = [0, 1/n)$. Each of the intervals

$$A_i = \left[\frac{i-1}{n}, \frac{i}{n}\right), \quad 2 \leq i \leq n,$$

has the same measure as $A = A_1$ and is disjoint from it. Thus

$$\mu(\widetilde{\Psi}(A_i)) = \mu(\widetilde{\Psi}(A)), \quad 2 \leq i \leq n.$$

Since the sets $\widetilde{\Psi}(A_i)$ are also pairwise disjoint and cover $\mathbb{T}^1$, it follows that $\mu(\widetilde{\Psi}(A_i)) = 1/n$. Consequently, $\widetilde{\Psi}$ preserves the measure of sets with rational measure. In general, the set A may be approximated by an increasing family of sets in $\mathcal{P}$ having rational measure. □

6.2 Proof of Theorem 1.6

Let Ψ be an arbitrary automorphism of $\mathcal{E}$. It will be shown that the identity automorphism may be reached by successively replacing Ψ with a composition of Ψ and an automorphism in $\langle \mathrm{Inn}(\mathcal{E}), \Psi_T \rangle$.

To begin, by Corollary 6.2, Ψ maps the rotation group R to a conjugate $\Phi_g(R)$. Replacing Ψ by $\Phi_g^{-1} \circ \Psi$, it may be assumed that R is invariant under Ψ. Let $\Psi_R : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ denote the restriction $\Psi|_R$, where $r_\alpha \mapsto \alpha$ is the natural identification of R and $\mathbb{R}/\mathbb{Z}$.

Lemma 6.7. *Ψ_R is continuous (w.r.t. the standard topology on $\mathbb{R}/\mathbb{Z}$).*

Proof. Since Ψ_R is a group homomorphism, it suffices to show that Ψ_R is continuous at $0 \in \mathbb{R}/\mathbb{Z}$. Suppose that $\alpha_n \to 0$ in $\mathbb{R}/\mathbb{Z}$. Then for any nonempty $A \in \mathcal{P}$, there exists a constant $M_A > 0$ such that

$$A \cap r_{\alpha_n}(A) \neq \emptyset, \quad \text{if } n \geq M_A.$$

Conversely, this condition characterizes sequences in $\mathbb{R}/\mathbb{Z}$ which converge to 0. In particular, given some sequence α_n, suppose that there exists a constant M_A as above for every nonempty $A \in \mathcal{P}$. For any $\epsilon > 0$, let $A_\epsilon = [0, \epsilon)$. Then

$$A_\epsilon \cap r_{\alpha_n}(A_\epsilon) \neq \emptyset, \quad \text{if } n \geq M_{A_\epsilon},$$

which implies that $|\alpha_n| < \epsilon$ for all $n \geq M_{A_\epsilon}$. Thus, $\alpha_n \to 0$.

Assuming again that $\alpha_n \to 0$, define $\beta_n = \Psi_R(\alpha_n)$, so $r_{\beta_n} = \Psi(r_{\alpha_n})$. Let $B \in \mathcal{P}$ be nonempty, and let $A = \widetilde{\Psi}^{-1}(B)$. Then by Proposition 6.6, part (2),

$$r_{\beta_n}(B) = \widetilde{\Psi}(r_\alpha(\widetilde{\Psi}^{-1}(B))) = \widetilde{\Psi}(r_{\alpha_n}(A)).$$

Consequently, $A \cap r_{\alpha_n}(A) \neq \emptyset$ if and only if $B \cap r_{\beta_n}(B) \neq \emptyset$. Therefore, if $\alpha_n \to 0$, then there exists M_B (namely, the M_A associated with α_n), such that

$$B \cap r_{\beta_n}(B) \neq \emptyset, \ \forall\, n \geq M_B.$$

From the above characterization of sequences converging to zero, it follows that Ψ_R is continuous at zero. □

The only continuous automorphisms of $\mathbb{R}/\mathbb{Z}$ are the identity and $x \mapsto -x$. Note that the restriction of the orientation-reversing automorphism Ψ_T induces the second of these automorphisms. Subsequently, after replacing Ψ by $\Psi \circ \Psi_T$ if Ψ_R is not the identity, it may be assumed that $\Psi_R = id$.

Lemma 6.8. *If $\Psi \in Aut(\mathcal{E})$ fixes the rotation group R, then $\widetilde{\Psi}$ maps any interval in $\mathcal{P}$ to another interval.*

Proof. Since any rotation will preserve intervals in $\mathcal{P}$, it suffices to consider $I_a = [0, a)$. Then there exists some $\epsilon > 0$, such that for any $\alpha \in (-\epsilon, \epsilon)$,

$$\mu(I_a \cap r_\alpha(I_a)) = a - |\alpha|.$$

By Proposition 6.6, part (2), and the hypothesis that $\Psi(r_\alpha) = r_\alpha$,

$$\widetilde{\Psi} \circ \widetilde{r}_\alpha = \widetilde{\Psi(r_\alpha)} \circ \widetilde{\Psi} = \widetilde{r}_\alpha \circ \widetilde{\Psi}.$$

Therefore, it is also the case that

$$\mu(\widetilde{\Psi}(I_a) \cap r_\alpha(\widetilde{\Psi}(I_a))) = a - |\alpha|,$$

for $\alpha \in (-\epsilon, \epsilon)$.

Suppose that $\widetilde{\Psi}(I_a)$ has $k \geq 1$ components:

$$\widetilde{\Psi}(I_a) = A_1 \cup \cdots \cup A_k,$$

where the A_i are pairwise disjoint intervals. Since the A_i are disjoint, there is some $\delta > 0$ such that

$$r_\beta(A_i) \cap A_j = \emptyset, \quad \text{for all } |\beta| < \delta \text{ and } i \neq j.$$

Consequently, if $|\beta| < \delta$, then

$$\mu(\widetilde{\Psi}(I_a) \cap r_\beta(\widetilde{\Psi}(I_a))) = a - k|\beta|.$$

It follows that $k = 1$, which implies that $\widetilde{\Psi}(I_a)$ must be an interval. $\square$

Continue with the assumption that Ψ_R is the identity. By the previous lemma, $\widetilde{\Psi}$ maps the interval $I_a = [0, a)$ to some translate of I_a. After composing Ψ with a suitable Φ_{r_β}, it may be assumed that Ψ_R is the identity and $\widetilde{\Psi}(I_a) = I_a$. Since

$$\widetilde{\Psi} \circ \widetilde{r}_\beta = \widetilde{r}_\beta \circ \widetilde{\Psi},$$

for all $\beta \in \mathbb{R}/\mathbb{Z}$, it follows that $\widetilde{\Psi}$ fixes any translate $\widetilde{r}_\beta(I_a) = [\beta, a+\beta)$. Thus, for any β, $0 < \beta < a$, $\widetilde{\Psi}$ fixes the intersection

$$I_a \cap r_\beta(I_a) = [\beta, a).$$

Thus $\widetilde{\Psi}$ fixes all translates of arbitrarily small intervals, which implies that $\widetilde{\Psi}$ is the identity on $\mathcal{P}$. Consequently, for any $f \in \mathcal{E}$, $\Psi(f)$ acts on the sets in $\mathcal{P}$ identically to the way f does, which implies Ψ is the identity. It has been shown that any $\Psi \in \mathrm{Aut}(\mathcal{E})$ is in the group $\langle \mathrm{Inn}(\mathcal{E}), \Psi_T \rangle$, and the proof of Theorem 1.6 is complete.

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